

A Sieve for Cousin Primes

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Abstract

A sieve is constructed for twin primes at distance 4, which are of the form $3(2m + 1) \pm 2$, and are characterized by their twin-4 rank $2m + 1$. It does not suffer from the parity problem. Non-rank numbers are identified and counted using odd primes $p \geq 5$. Twin-4 ranks and non-ranks make up the set of odd numbers. Regularities of non-ranks allow gathering information on them to obtain a Legendre-type sum for the number of twin-4 ranks. Due to considerable cancellations in it, the asymptotic law of its main term has the expected form and magnitude of its coefficient.

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1 Introduction

Our knowledge of twin primes comes mostly from sieve methods [1],[2],[3],[4].

In Ref. [5] a sieve is developed specifically for ordinary twin primes. These methods are applied here to cousin primes, where arithmetical details are rather different. However, there are also similarities because both their distances, 2 and 4, have no odd prime divisor.

Prime numbers $p \geq 5$ are well known to be of the form [6] $6m \pm 1$. An ordinary twin prime occurs when both $6m \pm 1$ are prime. Twin primes at distance 4 can be written similarly as $6m + 1, 6(m + 1) - 1$ or $3(2m + 1) \pm 2$, being in class II of a classification [7] of all twin primes, whereas ordinary twins lie in class I being of the form $2(3m) \pm 1$.

Definition 1.1. If $3(2m + 1) \pm 2$ is a twin prime pair for some odd $2m + 1$, then $2m + 1$ is called its *twin-4 rank*. An odd number $2m + 1$ is a *non-rank* if $3(2m + 1) \pm 2$ are not both prime. Odd positive integers ≥ 3 consist of twin-4- and non-ranks only. Even numbers are not considered in the following because $3(2m) \pm 2$ are never primes. Also, since 2, 3 are not of the form $6m \pm 1$, they are excluded as primes in the following. Also, we ignore the *special* cousin prime [7] $5 \pm 2 = (3, 7)$.

Example 1.2. Twin-4 ranks are 3, 5, 7, 13, 15, Non-ranks are 9, 11, 17, 19,

The odd numbers ≥ 3 form the base set of this pair sieve; it is partitioned into twin-4 and non-rank sets. Only non-ranks have sufficient regularity and abundance allowing us to gather enough information on them to draw inferences on the minimal number of twin-4 ranks needed to account for all odd numbers ≥ 3 . Therefore, our main focus is on non-ranks, their symmetries and abundance.

In Sect. 2 the twin-4 prime sieve is constructed based on non-ranks. In Sect. 3 non-ranks are identified in terms of their main properties and then, in Sect. 4, they are counted. In Sect. 5 twin-4 ranks are isolated and counted. Conclusions are summarized and discussed in Sect. 6.

2 Twin Ranks, Non-Ranks and Sieve

It is our goal here to construct a twin-4 prime sieve. To this end, we need the following arithmetical function.

Definition 2.1. Let x be real. Then $N(x)$ is the integer nearest to x . The ambiguity for $x = n + \frac{1}{2}$ with integral n will not arise in the following.

Lemma 2.2. Let $p \geq 5$ be prime. Then

$$N\left(\frac{p}{6}\right) = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}; \\ \frac{p+1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases} \quad (1)$$

Proof. This is obvious from Def. 2.1 by substituting $p = 6m \pm 1$. \diamond

Corollary 2.3. *If $p \equiv -1 \pmod{6}$ is prime and $p - 4$ is prime, then $\frac{p+1}{3} - 1$ is a twin-4 rank. If $p \equiv 1 \pmod{6}$ and $p + 4$ is prime, then $\frac{p-1}{3} + 1$ is a twin-4 rank.*

Proof. This follows from Def. 1.1, as $3(\frac{p+1}{3} - 1) \pm 2 = (p, p - 4)$, and $3(\frac{p-1}{3} + 1) \pm 2 = (p, p + 4)$ in the last case. \diamond

Example 2.4. This is the case for $p = 7, 13, 19, \dots$ as well as for $p = 11, 17, 23, \dots$ but not for $p = 5, 29, 31, \dots$

Lemma 2.5 *Let $p \geq 5$ be prime. Then all odd numbers*

$$\begin{aligned} 2k(n, p)^+ + 1 &= (2n + 1)p + 4N\left(\frac{p}{6}\right), \quad n = 0, 1, 2, \dots \\ 2k(n, p)^- + 1 &= (2n + 1)p - 4N\left(\frac{p}{6}\right), \quad n = 1, 2, \dots \end{aligned} \quad (2)$$

are non-ranks. There are $2 = 2^{\nu(p)}$ (single) non-rank progressions to the prime p , where $\nu(n)$ counts the number of different prime divisors of n .

(a) *If $p \equiv 1 \pmod{6}$ the non-rank $2k(n, p)^+ + 1$ generates the pair*

$$3(2k(n, p)^+ + 1) \pm 2 = ([3(2n + 1) + 2]p - 4, [3(2n + 1) + 2]p), \quad (3)$$

and the non-rank $2k(n, p)^- + 1$ the pair

$$3(2k(n, p)^- + 1) \pm 2 = ([3(2n + 1) - 2]p, [3(2n + 1) - 2]p + 4). \quad (4)$$

(b) *If $p \equiv -1 \pmod{6}$ the non-rank $2k(n, p)^+ + 1$ generates the pair*

$$3(2k(n, p)^+ + 1) \pm 2 = ([3(2n + 1) + 2]p - 4, [3(2n + 1) + 2]p), \quad (5)$$

and the non-rank $2k(n, p)^- + 1$ the pair

$$3(2k(n, p)^- + 1) \pm 2 = ([3(2n + 1) - 2]p - 4, [3(2n + 1) - 2]p). \quad (6)$$

All pairs contain a composite number.

For $n = 0$ and $p \equiv \pm 1 \pmod{6}$, $2k^\pm + 1 \rightarrow \frac{p \pm 2}{3}$ are the twin-4 ranks of Cor. 2.3.

Clearly, all these non-ranks are symmetrically distributed at equal distances $4N(p/6)$ from odd multiples of each prime $p \geq 5$. Twin-4 ranks and some non-ranks for $n = 0$ are the subject of Cor. 2.3 and Example 1.2.

Proof. Let $p \equiv 1 \pmod{6}$ be prime and $n \geq 0$ an integer. Then $2k(n, p)^+ + 1 = (2n + 1)p + 4\frac{p-1}{6}$ by Lemma 2.2 and $3(2k^+ + 1)$ is sandwiched by the pair in Eq. (3) which contains a composite number. Hence $2k(n, p)^+ + 1$

is a non-rank. For $n > 0$, the same happens in Eq. (4), so $2k^- + 1$ is a non-rank.

If $p \equiv -1 \pmod{6}$ and prime, then $2k(n, p)^+ + 1 = (2n + 1)p + 4\frac{p+1}{6}$ by Lemma 2.2 and $3(2k^+ + 1)$ leads to the pair in Eq. (5) which contains a composite number again. For $n > 0$, the same happens in Eq. (6), so $2k^- + 1$ is a non-rank. \diamond

The $2k(n, p)^\pm + 1$ yield pairs $3(2k^\pm + 1) \pm 2$ with one or two composite entries that are twin-4 prime analogs of multiples np , $n > 1$ of a prime p in Eratosthenes' prime sieve [6].

The converse of Lemma 2.5 holds, i.e. non-ranks are organized in arithmetic progressions by prime numbers ≥ 5 .

Lemma 2.6. *If $2k + 1 > 3$ is a non-rank, there is a prime $p \geq 5$ and a non-negative odd integer $2\kappa \pm 1$ so that $2k + 1 = 2k(\kappa, p)^+ + 1$ or $2k(\kappa, p)^- + 1$.*

Proof. Let $6k + 5 = 3(2k + 1) + 2$ be composite. Then $3(2k + 1) + 2 = 6k + 5 \neq 2^\mu 3^\nu$, $\mu + \nu \geq 1$ obviously. Let $6k + 5 = p \cdot K \equiv -1 \pmod{6}$, where $p \geq 5$ is the smallest prime divisor. If $p = 6m + 1$, then $K = 6\kappa - 1$ and

$$6k + 5 = 6^2 m \kappa + 6(\kappa - m) - 1, \quad k + 1 = 6m\kappa + \kappa - m = p\kappa - \frac{p-1}{6} \quad (7)$$

and

$$2k + 1 = 2p\kappa - \frac{p-1}{3} - 1 = (2\kappa - 1)p + 4\frac{p-1}{6}, \quad (8)$$

q.e.d. If $p = 6m - 1$, then $K = 6\kappa + 1$ and

$$6k + 5 = 6^2 m \kappa + 6(m - \kappa) - 1, \quad k + 1 = 6m\kappa + m - \kappa = p\kappa + \frac{p+1}{6} \quad (9)$$

and so

$$2k + 1 = 2p\kappa + \frac{p+1}{3} - 1 = (2\kappa + 1)p - 4\frac{p+1}{6}, \quad (10)$$

q.e.d. Now let $6k + 1 = pK$ with $p \equiv 1 \pmod{6}$. Then $K = 6\kappa + 1$ and, therefore,

$$k = 6m\kappa + m + \kappa = p\kappa + \frac{p-1}{6}, \quad 2k + 1 = (2\kappa + 1)p - 4\frac{p-1}{6}, \quad (11)$$

q.e.d. Finally, if $p \equiv -1 \pmod{6}$ then $K = 6\kappa - 1$ and

$$\begin{aligned} 6k + 1 &= (6m - 1)(6\kappa - 1) = 6^2 m\kappa - 6(m + \kappa) + 1, \\ k &= 6m\kappa - (m + \kappa) = p\kappa - \frac{p + 1}{6}. \end{aligned} \quad (12)$$

Hence

$$2k + 1 = (2\kappa - 1)p + 4\frac{p + 1}{6}, \quad (13)$$

q.e.d. \diamond

Even multiples of prime numbers $p \geq 5$ in Lemma 2.5, e.g. $2np + 1 \pm 2\alpha N(p/6)$ for appropriate α , are accounted for in Lemma 2.6 as non-ranks to some prime $p' \geq 5$, which demonstrates the cornerstone role Lemma 2.6 plays for the sieve.

Theorem 2.7. (Cousin Prime Sieve) *Let $\mathcal{P} = \{(2n + 1 \geq 3, 2n + 5) : n \geq 0, \text{ integral}\}$ be the set of all pairs with entries ≥ 3 of natural numbers at distance 4. Upon striking all pairs identified by non-ranks of Lemma 2.5, only (and all) twin-4 prime pairs are left.*

Clearly, this sieve is not subject to the parity problem.

Proof. Obviously, we need to consider only the subset $\mathcal{P}_0 = \{(6m + 1, 6m + 5) : m \geq 0, \text{ integral}\} \subset \mathcal{P}$. For $2m + 1 \geq 3$ divide $3(2m + 1) \pm 2$ by all primes $p < \sqrt{6m + 1}$. Then $2m + 1$ is a non-rank if there is a prime p such that $(6m + 5)/p$ or $(6m + 1)/p$ (or both) is integral. For all such m , $2m + 1$ is struck from the set of odd positive integers. Then all remaining odd integers are twin-4 ranks. \diamond

More concrete steps to construct it will be taken in the next section.

3 Identifying Non-Ranks

Here it is our goal to systematically characterize and identify non-ranks among odd numbers.

Definition 3.1 Let $p \geq 5$ be the minimal prime of a non-rank. Then p is its parent prime.

Example 3.2. The non-ranks to parent prime 5 are, by Lemma 2.5,

$$2k^+ + 1 = 9, 19, 29, \dots; \quad 2k^- + 1 = 11, 21, \dots \quad (14)$$

These $2k^\pm + 1$ form the set $\mathcal{A}_5^- = \{5(2n+1) \pm 4 \geq 9 : n \geq 0\} = \mathcal{A}_5$. Note that 5 is the most effective non-rank generating prime number. If it were excluded like 3 then many numbers, such as 9, 19, ..., would be missed as non-ranks.

Proposition 3.3. *The arithmetic progressions $3 \cdot 5(2n+1) \pm 2, 3[5(2n+1) + 2] \pm 2, 3[5(2n+1) - 2] \pm 2$ contain all twin-4 prime pairs.*

Prop. 3.3 is the first step of the twin-4-prime sieve. Let $\mathcal{C}_5 = \{0, 2, 8\}$ be the set of non-negative constants c in $5(2n+1) + c$ in Prop. 3.3.

Proof. From $\{3 \cdot 5(2n+1) \pm 2, 3[5(2n+1) + 2] \pm 2, 3[5(2n+1) - 2] \pm 2, 3[5(2n+1) + 4] \pm 2, 3[5(2n+1) - 4] \pm 2, n \geq 0\}$ we strike all pairs $\{3 \cdot 5(2n+1) + 12 \pm 2, 3 \cdot 5(2n+1) - 12 \pm 2, n > 0\}$ resulting from non-ranks of \mathcal{A}_5^- . \diamond

For $p = 7$, we now subtract from the set $\mathcal{A}_7^+ = \{7(2n+1) \pm 4 \geq 9 : n \geq 0\}$ of non-ranks to 7 those to $p = 5$. The remaining set \mathcal{A}_7 comprises the non-ranks to parent prime $p = 7$.

Lemma 3.4. *The set \mathcal{A}_7 of non-ranks to parent prime $p = 7$ comprises the arithmetic progressions $\{7 \cdot 5(2n+1) + 10, 7 \cdot 5(2n+1) + 18, 7 \cdot 5(2n+1) + 32, 7 \cdot 5(2n+1) + 38, 7 \cdot 5(2n+1) + 52, 7 \cdot 5(2n+1) + 60\}$.*

Proof. We subtract the common arithmetic progressions of $\mathcal{A}_5^- : \{5(2n+1) \pm 4 : 2n+1 \rightarrow 7(2n+1), 7(2n+1) \pm 2, 7(2n+1) \pm 4, 7(2n+1) \pm 6\}$ from $\mathcal{A}_7^+ : \{7(2n+1) \pm 4 : 2n+1 \rightarrow 5(2n+1), 5(2n+1) \pm 2, 5(2n+1) \pm 4\}$ to find those listed in Lemma 3.4.

The common arithmetic progressions are

$$\begin{aligned} &5 \cdot 7(2n+1) \pm 4, \\ &5[7(2n+1) + 4] + 4 = 7[5(2n+1) + 4] - 4, \\ &5[7(2n+1) - 4] - 4 = 7[5(2n+1) - 4] + 4. \quad \diamond \end{aligned} \tag{15}$$

Note that these $2^{\nu(5 \cdot 7)} = 2^2$ arithmetic progressions contain all common (double) non-ranks of the primes 5, 7.

Proposition 3.5. *The arithmetic progressions $3[5 \cdot 7(2n+1) + c] \pm 2, n \geq 0$ contain all twin-4 pairs ≥ 103 , where $c \in \mathcal{C}_7 = \{0, 2, 8, 12, 20, 22, 28, 30, 40, 42, 48, 50, 58, 62, 68\}$.*

Note that $\mathcal{C}_5 \subset \mathcal{C}_7$, but this pattern does not continue.

Proof. Using Lemma 3.4, we strike from the arithmetic progressions of Prop. 3.3 (replacing $2n+1 \rightarrow 7(2n+1), 7(2n+1) \pm 2, 7(2n+1) \pm 4, 7(2n+1) \pm 6$) all pairs resulting from non-ranks in \mathcal{A}_7 , which are $\{5 \cdot 7(2n+1) + a; a = 10, 18, 32, 38, 52, 60\}$. This leaves the progressions listed above. \diamond

This is the second step of the sieve.

In contrast to ordinary twin primes the arithmetic function values $N(p'/6)$, $N(p/6)$ do not suffice to characterize twin-4 primes $p' = p + 4$.

Theorem 3.6. *Let p', p be primes. If $p' \equiv -1 \pmod{6}$, $p \equiv 1 \pmod{6}$ and $N(\frac{p'}{6}) = 1 + N(\frac{p}{6})$ then $p' = p + 4$.*

Proof. If $p' \equiv -1 \pmod{6}$, $p \equiv 1 \pmod{6}$ then

$$\frac{p' + 1}{6} = 1 + \frac{p - 1}{6} \quad (16)$$

is equivalent to $p' = p + 4$. \diamond

Corollary 3.7. *Let $p' > p \geq 5$ be primes such that $N(\frac{p'}{6}) = 1 + N(\frac{p}{6})$. Then $p' = p + 4$ if $p \equiv 1 \pmod{6}$ and $p' \equiv -1 \pmod{6}$; if $p' \equiv 1 \pmod{6}$ instead then $p' = p + 6$. If $p \equiv -1 \pmod{6}$ and $p' \equiv -1 \pmod{6}$ then $p' = p + 6$; if $p' \equiv 1 \pmod{6}$ instead then $p' = p + 8$.*

Proof. Let $p \equiv -1 \pmod{6}$. Then $1 + N(\frac{p}{6}) = 1 + \frac{p+1}{6}$. If $p' \equiv -1 \pmod{6}$ then $p' = p + 6$. If $p' \equiv 1 \pmod{6}$ then $p' = p + 8$. If $p \equiv 1 \pmod{6}$ then $1 + \frac{p-1}{6} = N(p'/6)$ implies $p' = p + 6$ if $p' \equiv 1 \pmod{6}$; if $p' \equiv -1 \pmod{6}$ then $p' = p + 4$. \diamond

Theorem 3.8. *Let $p \geq 5$, $p' = p + 2$ be ordinary prime twins. Then $pp'(2n + 1) \pm 4\frac{p+1}{6} > 0$ for $n = 0, 1, 2, \dots$ and*

$$\begin{aligned} p[p'(2n + 1) + 4\frac{p+1}{6}] + 4\frac{p+1}{6} &= p'[p(2n + 1) + 4\frac{p+1}{6}] - 4\frac{p'-1}{6} > 0, \\ p[p'(2n + 1) - 4\frac{p+1}{6}] - 4\frac{p+1}{6} &= p'[p(2n + 1) - 4\frac{p+1}{6}] + 4\frac{p'-1}{6} > 0, \\ n = 0, 1, 2, \dots \end{aligned} \quad (17)$$

are their common non-ranks.

Note that again there are 4 arithmetic progressions of common or double non-ranks.

Proof. Using $N(p'/6) = N(p/6) = \frac{p+1}{6}$, Eq. (17) is readily verified; its lhs $\in \mathcal{A}_p^-$ and rhs $\in \mathcal{A}_p^+$ and $p(p+2)(2n+1) \pm 4\frac{p+1}{6} \in \mathcal{A}_p^-, \mathcal{A}_p^+$. \diamond

We now consider systematically common non-ranks of pairs of primes generalizing Theor. 3.8 to arbitrary prime pairs.

Theorem 3.9. *Let $p' > p \geq 5$ be primes. (i) If $p' \equiv p \equiv -1 \pmod{6}$, then $p' = p + 6l$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$ and common non-ranks of p', p are, for $n = 0, 1, \dots$,*

$$p[p'(2n + 1) + 2r] \pm 4N(\frac{p}{6}) = p'[p(2n + 1) + 2r] \pm 4N(\frac{p'}{6}) \quad (18)$$

provided the integers r, r' solve

$$(r' - r)p = 2l(3r \pm 1), \quad 1 - p \leq 2r \leq p - 1, \quad 1 - p' \leq 2r' \leq p' - 1. \quad (19)$$

Eq. (19) with $3r \pm 1 \equiv 0 \pmod{p}$ on the rhs has a unique solution r that then determines r' .

If r, r' solve

$$(r' - r)p = 2l(3r \mp 1)l \mp 2N\left(\frac{p+1}{3}\right) \quad (20)$$

then the common non-ranks are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \mp 4N\left(\frac{p'}{6}\right). \quad (21)$$

(ii) If $p' \equiv p \equiv 1 \pmod{6}$, then $p' = p + 6l$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and common non-ranks of p', p are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \pm 4N\left(\frac{p'}{6}\right) \quad (22)$$

provided r, r' solve

$$(r' - r)p = 2l(3r \pm 1). \quad (23)$$

If r, r' solve

$$(r' - r)p = 2l(3r \mp 1)l \mp 2N\left(\frac{p-1}{6}\right) \quad (24)$$

then the common non-ranks are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \mp 4N\left(\frac{p'}{6}\right). \quad (25)$$

(iii) If $p' \equiv 1 \pmod{6}$, $p \equiv -1 \pmod{6}$ then $p' = p + 6l + 2$, $l \geq 0$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and common non-ranks of p', p are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \pm 4N\left(\frac{p'}{6}\right) \quad (26)$$

provided

$$(r' - r)p = 2r(3l + 1) \pm 2l. \quad (27)$$

If $l = 0$ then $r' = r = 0$ and Eq. (17) are solutions (Cor. 3.7).

If r, r' solve

$$(r' - r)p = 2r(3l + 1)l \mp 2 \left(l + \frac{p+1}{3} \right), \quad l \geq 1, \quad (28)$$

then the common non-ranks are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \mp 4N\left(\frac{p'}{6}\right). \quad (29)$$

(iv) If $p' \equiv -1 \pmod{6}$, $p \equiv 1 \pmod{6}$ then $p' = p + 6l - 2$, $l \geq 1$, $N\left(\frac{p'}{6}\right) = N\left(\frac{p}{6}\right) + l$, and common non-ranks of p', p are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \pm 4N\left(\frac{p'}{6}\right) \quad (30)$$

provided

$$(r' - r)p = 2r(3l - 1) \pm 2l. \quad (31)$$

If r, r' solve

$$(r' - r)p = 2r(3l - 1) \mp 2 \left(l + \frac{p-1}{3} \right) \quad (32)$$

then the common non-ranks are

$$p[p'(2n+1) + 2r'] \pm 4N\left(\frac{p}{6}\right) = p'[p(2n+1) + 2r] \mp 4N\left(\frac{p'}{6}\right). \quad (33)$$

Note that, again, there are $4 = 2^{\nu(pp')}$ arithmetic progressions of common or double non-ranks to the primes p', p in all cases.

Proof. By substituting $p', N(p'/6)$ in terms of $p, N(p/6)$ and l , respectively, it is readily verified that Eqs. (18), (19) are equivalent, as are (20), (21), and (22), (23), and (24), (25), and (26), (27), and (28), (29), and (30), (31), and (32), (33). As in (i) there is a unique solution (r, r') in all other cases as well. \diamond

Example 3.10. For $p = 5$, $p' = 11$ we have $l = 1$ and Eq. (19) becomes $5(r' - r) = 2(3r + 1)$, i.e. $r = 3$, $r' - r = 4$ or $r' = 7$. So Eq. (18) gives the common non-ranks

$$5[11(2(n+1) + 1) - 8] + 4 = 11[5(2(n+1) + 1) - 4] + 4 \cdot 2. \quad (34)$$

The other sign in Eq. (19) is $5(r' - r) = 2(3r - 1)$ solved by $r = 2$, $r' - r = 2$, $r' = 4$ and the common non-ranks are

$$5[11(2n + 1) + 8] - 4 = 11[5(2n + 1) + 4] - 4 \cdot 2. \quad (35)$$

Theorem 3.11. (Triple non-ranks) *Let $5 \leq p < p' < p''$ (or $5 \leq p < p'' < p'$, or $5 \leq p'' < p < p'$) be different odd primes. Then each case in Theor. 3.9 of four double non-ranks leads to $8 = 2^{\nu(pp'p'')}$ triple non-ranks of p, p', p'' . At two non-ranks per prime, there are at most 2^3 triple non-ranks.*

Proof. It is based on Theor. 3.9 and similar for all its cases. Let's take (i) and substitute $2n + 1 \rightarrow p''(2n + 1) + 2\nu$, $-p'' < 2\nu < p''$ in Eq. (18) which, upon dropping the term $p''p'p(2n + 1)$, yields on the lhs

$$2pp'\nu + 2pr' - 4N\left(\frac{p}{6}\right) = 2p''\mu \pm 4N\left(\frac{p''}{6}\right). \quad (36)$$

Since $(pp', p'') = 1$ there is a unique residue ν modulo p'' so that the lhs of Eq. (36) is $\equiv \pm 4N\left(\frac{p''}{6}\right) \pmod{p''}$, and this determines μ . As each sign case leads to such a triple non-rank solution, it is clear that there are 2^3 non-ranks to p, p', p'' . \diamond

Example 3.12. For 5, 7, 11 triple non-rank progressions are obtained as follows. Starting from the double non-rank equations (cf. proof of Lemma 3.4)

$$5 \cdot 7(2n + 1) \pm 4, \quad 5[7(2n + 1) + 4] + 4 = 7[5(2n + 1) + 4] - 4 \quad (37)$$

replace $2n + 1 \rightarrow 11(2n + 1) + 2\nu$, drop $5 \cdot 7 \cdot 11(2n + 1)$ and set the rhs to $22\mu + 4N(11/6)$:

$$5 \cdot 7 \cdot 2\nu + 4 = 11 \cdot 2\mu + 8. \quad (38)$$

Since $5 \cdot 7 \cdot 2 - 4 = 11 \cdot 6$, the solution is $\nu = 1$, $\mu = 3$. Putting back $5 \cdot 7 \cdot 11(2n + 1)$ we obtain the triple non-rank system

$$\begin{aligned} 5 \cdot 7[11(2n + 1) + 2] + 4 &= 7[5 \cdot 11(2n + 1) + 10] + 4 \\ &= 11[5 \cdot 7(2n + 1) + 6] + 2 \cdot 4. \end{aligned} \quad (39)$$

Setting the rhs to $22\mu - 8$ yields the second such solution

$$35 \cdot 2\nu + 4 = 22\mu - 8, \quad \nu = 4, \quad \mu = 13, \quad (40)$$

with the complete triple non-rank system

$$\begin{aligned} 5 \cdot 7[11(2(n+1)+1)-6] + 4 &= 7 \cdot 5[11(2(n+1)+1)-6] + 4 \\ &= 11[5[7(2(n+1)+1)-4] + 2] - 4 \cdot 2. \end{aligned} \quad (41)$$

Theorem 3.13. (Multiple non-ranks) *Let $5 \leq p_1 < \dots < p_m$ be m different primes. Then there are 2^m arithmetic progressions of m -fold non-ranks to the primes p_1, \dots, p_m .*

Proof. This is proved by induction on m . Theors. 3.9 and 3.11 are the $m = 2, 3$ cases. If Theor. 3.13 is true for m then for any case $5 \leq p_{m+1} < p_1 < \dots < p_m$, or \dots , $5 \leq p_1 < \dots < p_{m+1}$, we substitute in an m -fold non-rank equation $2n+1 \rightarrow p_{m+1}(2n+1)+2\nu$ as in the proof of Theor. 3.11, again dropping the $(2n+1) \prod_{i=1}^{m+1} p_i$ term. Then we get

$$\begin{aligned} p_1(p_2(\dots(2p_m\nu + 2r_m) + \dots + 2r_2) + 4N(\frac{p_1}{6}) \\ = 2p_{m+1}\mu \pm 4N(\frac{p_{m+1}}{6}) \end{aligned} \quad (42)$$

with a unique residue $2\nu \pmod{p_{m+1}}$ so that the lhs of Eq. (42) becomes $\equiv 4N(\frac{p_{m+1}}{6}) \pmod{p_{m+1}}$, which then determines 2μ . In case the lhs of Eq. (42) has $p_1(\dots) - 4N(p_1/6)$ the argument is the same. This yields an $(m+1)$ -fold non-rank progression since each sign in Eq. (42) gives a solution. Hence there are 2^{m+1} such non-ranks. At two non-ranks per prime there are at most 2^m non-rank progressions. \diamond

4 Counting Non-Ranks

If we subtract for case (i) in Theor. 3.9, say, the four common non-rank progressions this leaves in $\mathcal{A}_{p'}^- = \{p'(2n+1) \pm 4\frac{p'+1}{6} : n \geq 0\}$ the following progressions $p'p(2n+1) \pm 4\frac{p'+1}{6}, \dots, p'[(2n+1)p + 2r_1] + 4\frac{p'+1}{6}, \dots, p'[(2n+1)p + 2r_2] - 4\frac{p'+1}{6}, \dots, p'[(2n+1)p + 2r_3] + 4\frac{p'+1}{6}, \dots, p'[(2n+1)p + 2r_4] - 4\frac{p'+1}{6}, \dots, p'(2n+1)p \pm 4\frac{p'+1}{6}$.

We summarize this as follows.

Lemma 4.1. *$p' > p \geq 5$ be prime. Removing the common non-ranks of p', p from the set of all non-ranks of p' leaves arithmetic progressions of the form $p'(2n+1)p + 2l$; $n \geq 0$ where $l > 0$ are given non-negative integers.*

Proposition 4.2. *Let $p > 5$ be prime. Then the set of non-ranks to parent prime p , \mathcal{A}_p , is made up of arithmetic progressions $L(p)(2n+1) + 2a$, $n \geq 0$ with $L(p) = \prod_{5 \leq p' \leq p} p$, p' prime and $a > 0$ given integers.*

Proof. Let $p = 6m \pm 1$. We start from the set $\mathcal{A}_p^\pm = \{p(2n+1) \pm 4N(\frac{p}{6}) > 0 : n = 0, 1, 2, \dots\}$. Removing the non-ranks common to p and 5 by Lemma 4.1 leaves arithmetic progressions of the form $5p(2n+1) + 2l$, $n \geq 0$ where $l > 0$ are given integers. Continuing this process to the largest prime $p' < p$ leaves in \mathcal{A}_p arithmetic progressions of the form $L(p)(2n+1) + 2a$, $n \geq 0$ with $L(p) = \prod_{5 \leq p' \leq p} p'$ and $a > 0$ a sequence of given integers independent of n . \diamond

Proposition 4.3. *Let $p \geq p' \geq 5$ be primes and $G(p)$ the number of non-ranks $L(p)(2n+1) + 2a \in \mathcal{A}_p$ over one period $L(p)$ corresponding to arithmetic progressions $L(p)(2n+1) + 2a \in \mathcal{A}_p$. Then $G(p) = \prod_{5 \leq p' < p} (p' - 2)$.*

Note that $G(p) < L(p)$ both increase monotonically as $p \rightarrow \infty$.

Proof. In order to determine $G(p)$ we have to eliminate all non-ranks of primes $5 \leq p' < p$ from \mathcal{A}_p . As in Lemma 3.4 we start by subtracting the fraction $2/5$ from the interval $1 \leq a \leq L(p)$ of length $L(p)$, then $2/7$ for $p' = 7$ and so on for all $p' < p$. The factor of 2 is due to the symmetry of non-ranks around each multiple of p' according to Lemma 2.5. This leaves $p \prod_{5 \leq p' < p} (p' - 2)/2$ odd numbers. The fraction $2/p$ of these are the non-ranks to parent prime p . \diamond

Prop. 4.3 implies that the fraction of non-ranks related to a prime p in the interval occupied by \mathcal{A}_p ,

$$q(p) = \frac{G(p)}{L(p)} = \frac{1}{p} \prod_{5 \leq p' < p} \frac{p' - 2}{p'}, \quad (43)$$

where p' is prime, decreases monotonically as p goes up.

Definition 4.4. Let $p \geq p' \geq 5$ be prime. The supergroup $\mathcal{S}_p = \bigcup_{p' \leq p} \mathcal{A}_{p'}$ contains the sets of non-ranks corresponding to arithmetic non-rank progressions $2a + L(p')(2n+1)$ of all $\mathcal{A}_{p'}$, $p' \leq p$.

Thus, each supergroup \mathcal{S}_p contains nested sets of non-ranks related to primes $5 \leq p' \leq p$.

Let us now count prime numbers from $p_1 = 2$ on.

Proposition 4.5. *Let $p_j \geq 5$ be the j th prime. (i) Then the number of non-ranks $a \in \mathcal{A}_{p_i}$ corresponding to arithmetic progressions related to a*

prime $5 \leq p_i < p_j$,

$$G(p_i) = \frac{L(p_j)}{L(p_i)} G(p_j) = \frac{L(p_j)}{p_i} \prod_{5 \leq p < p_i} \frac{p-2}{p} = q(p_i) L(p_j), \quad (44)$$

where p is prime, monotonically decreases as p_i goes up. (ii) The number of non-ranks in a supergroup \mathcal{S}_{p_j} over one period $L(p_j)$ is

$$S(p_j) = L(p_j) \sum_{5 \leq p \leq p_j} q(p) = \frac{1}{2} L(p_j) \left(1 - \prod_{5 \leq p \leq p_j} \frac{p-2}{p} \right). \quad (45)$$

(iii) The fraction of non-ranks of their arithmetic progressions in the (first) interval $[1, L(p_j)]$ occupied by the supergroup \mathcal{S}_{p_j} ,

$$Q(p_j) = \frac{S(p_j)}{L(p_j)} = \sum_{5 \leq p \leq p_j} q(p) = \frac{1}{2} \left[1 - \prod_{5 \leq p \leq p_j} \frac{p-2}{p} \right], \quad (46)$$

increases monotonically as p_j goes up.

Proof. (i) follows from Prop. 4.3 and Eq. (43). (ii) and (iii) are equivalent and are proved by induction as follows, using Def. 4.4 in conjunction with Eq. (43).

From Eq. (43) we get $q_3 = 2/p_3$ which is the case $j = 3$, $p_j = 5$ of Eq. (46). Assuming Eq. (46) for p_j , we add q_{j+1} of Eq. (43) and obtain

$$\begin{aligned} \sum_{i=3}^{j+1} q(p_i) &= \frac{1}{2} - \frac{1}{2} \prod_{i=3}^j \frac{p_i-2}{p_i} + \frac{1}{p_{j+1}} \prod_{i=3}^j \frac{p_i-2}{p_i} \\ &= \frac{1}{2} - \frac{1}{2} \prod_{i=3}^{j+1} \frac{p_i-2}{p_i}. \end{aligned} \quad (47)$$

The extra factor $0 < (p_{j+1} - 2)/p_{j+1} < 1$ shows that $q(p_j), x(p_j)$ in Eq. (49) decrease monotonically as $p_j \rightarrow p_{j+1}$ while $Q(p_j)$ increases as $j \rightarrow \infty$. \diamond

Definition 4.6. Since $L(p) > S(p)$, there is a set \mathcal{R}_p of *remnants* $r \in [1, L(p)]$ such that $r \notin \mathcal{S}_p$.

Lemma 4.7. (i) The number $R(p_j)$ of remnants in a supergroup, \mathcal{S}_{p_j} , is

$$R(p_j) = \frac{1}{2} L(p_j) - S(p_j) = L(p_j) \left(\frac{1}{2} - Q(p_j) \right) = \frac{1}{2} \prod_{5 \leq p \leq p_j} (p-2) = \frac{1}{2} G(p_{j+1}). \quad (48)$$

(ii) The fraction of remnants in \mathcal{S}_{p_j} ,

$$x(p_j) = \frac{R(p_j)}{L(p_j)} = \frac{1}{2} - Q(p_j) = \frac{1}{2} \prod_{5 \leq p \leq p_j} \frac{p-2}{p}, \quad (49)$$

where p is prime, decreases monotonically as $p_j \rightarrow \infty$.

Proof. (i) follows from Def. 4.6 in conjunction with Eq. (45) and (ii) from Eq. (48). Eq. (48) follows from Eq. (46). \diamond

5 Remnants and Twin Ranks

When all primes $5 \leq p \leq p_j$ and appropriate nonnegative integers n are used in Lemma 2.5 one will find all non-ranks $2k+1 < M(j+1) \equiv (p_{j+1}^2 - 2^2)/3$. By subtracting these non-ranks from the set of odd positive integers $N \leq M(j+1)$ all and only twin ranks $t < M(j+1)$ are left among the remnants. If a non-rank $2k+1$ is left then $3(2k+1) \pm 2$ must have prime divisors that are $> p_j$ according to Lemma 2.5, which is impossible. All $t < M(j+1) = (p_{j+1}^2 - 4)/3$ in a remnant \mathcal{R}_{p_j} of a supergroup \mathcal{S}_{p_j} are twin-4 ranks.

Proposition 5.1. *Let p_j be the j th prime number and $L(p_j)(2n+1) + a_i^{(j)}$ be the arithmetic progressions that contain the non-ranks $a_i^{(j)} \in \mathcal{A}_{p_j}$ to parent prime p_j . Let $3[L(p_j)(2n+1) + c_i^{(j)}] \pm 2$ be the arithmetic progressions that contain the twin-4 primes with $c_i^{(j)} \in \mathcal{C}_{p_j}$.*

(i) *The set of constants $c_i^{(j+1)}$ of arithmetic progressions containing the twin-4 ranks from the next supergroup $\mathcal{S}_{p_{j+1}}$ is*

$$\begin{aligned} \mathcal{C}_{p_{j+1}} = & \{3[L(p_j)(p_{j+1}(2n+1) + l) + c_i^{(j)}] \pm 2\} \\ & - \{3[L(p_j)(p_{j+1}(2n+1) + l') + a_{i'}^{(j)}] \pm 2\}. \end{aligned} \quad (50)$$

If there are positive integers $0 \leq l, l' < p_{j+1}$, a non-rank $a_{i'}^{(j)} \in \mathcal{A}_{p_j}$ and a constant $c_i^{(j)} \in \mathcal{C}_{p_j}$ satisfying

$$L(p_j)l + c_i^{(j)} = L(p_j)l' + a_{i'}^{(j)}, \quad (51)$$

then

$$L(p_j)l + c_i^{(j)} \notin \mathcal{C}_{p_{j+1}}, \quad (52)$$

else

$$c_{i,l}^{(j+1)} = L(p_j)l + c_i^{(j)} \in \mathcal{C}_{p_{j+1}}. \quad (53)$$

Prop. 5.1 is the inductive step completing the practical sieve construction for ordinary twin primes. Props. 3.3, 3.5 and Lemma 3.4 are the initial steps.

Proof. Replacing in (i) $2n+1 \rightarrow p_{j+1}(2n+1)+l$, $l = 0, 1, 2, \dots, p_{j+1}-1$ and subtracting the resulting sets from each other, we obtain (i). \diamond

For $p_3 = 5$, Prop. 5.1 is Prop. 3.3, for $p_4 = 7$ it is Prop. 3.5. Clearly, at the start of the c for $p_4 = 7$ the previous values for $p_3 = 5$ are repeated, but this pattern does not continue.

Twin-4 ranks are located among the remnants \mathcal{R}_p for any prime $p \geq 5$. Our goal is to develop a Legendre-type sum for the number R of twin-4 ranks.

Theorem 5.2. *Let R_0 be the number of remnants of the supergroup \mathcal{S}_{p_j} , where p_j is the j th prime number and $M(j+1) = [p_{j+1}^2 - 4]/3$. Then the number $R = \pi_2(3L(p_j) + 2)/2$ of twin-4 ranks within the remnants of the supergroup \mathcal{S}_{p_j} is given by*

$$R = R_0 + \sum_{p_j < n} \mu(n) 2^{\nu(n)} \left[\frac{L(p_j) - M(j+1) - 1}{2n} \right]. \quad (54)$$

Here $L(p_j) = \prod_{5 \leq p \leq p_j} p$, $R_0 = \frac{1}{2} \prod_{5 \leq p \leq p_j} (p-2)$ with p prime, and n runs through all products of primes $p_j < p \leq (6L(p_j) + 1)/4$. The upper limit $(6L(p_j) + 1)/4$ comes about because $4N(p/6)$ is the lowest non-rank of the prime number p according to Lemma 2.2.

The argument of the twin-prime counting function π_2 is $3L(p_j) + 2$ because, if $L(p_j)$ is the last twin-4 rank of the interval $[1, L(p_j)]$, then $3L(p_j) \pm 2$ are the corresponding twin-4 primes.

Proof. According to Prop. 4.5 the supergroup \mathcal{S}_{p_j} has $S(p_j) = \frac{1}{2}L(p_j) \cdot \left(1 - \prod_{5 \leq p \leq p_j} \frac{p-2}{p}\right)$ non-ranks. Subtracting these from the interval $[1, L(p_j)]$ that the supergroup occupies gives $R_0 = \frac{1}{2} \prod_{5 \leq p \leq p_j} (p-2)$ for the number of remnants which include twin-4 ranks and non-ranks to primes $p_j < p \leq (6L(p_j) + 1)/4$. The latter are

$$M(j+1) < p(2n+1) \pm 4N\left(\frac{p}{6}\right) \leq L(p_j), \quad M(j+1) = \frac{p_{j+1}^2 - 4}{3}, \quad (55)$$

or

$$0 < n \leq \frac{L(p_j) - M(j+1) - 1}{2p}, \quad (56)$$

which have to be subtracted from the remnants to leave just twin-4 ranks. Correcting for double counting of common non-ranks to two primes using Theor. 3.9, of triple non-ranks using Theor. 3.11 and multiple non-ranks using Theor. 3.13 we obtain

$$\begin{aligned} R &= R_0 - 2 \sum_{p_j < p \leq (6L(p_j)+1)/4} \left[\frac{L(p_j) - M(j+1) - 1}{2p} \right] \\ &+ 4 \sum_{p_j < p < p' \leq (6L(p_j)+1)/4} \left[\frac{L(p_j) - M(j+1) - 1}{2pp'} \right] \mp \dots, \end{aligned} \quad (57)$$

where $[x]$ is the integer part of x as usual. Note that the arithmetic details (functions of r, r' in double non-ranks in Theor. 3.9, etc., that do not depend on n) do not affect their counting in Eq. (57) because they always add to $p'p(2n+1), pp'p''(2n+1), \dots$. Equation (57) is equivalent to Eq. (54). \diamond

Definition 5.3. Decomposing the floor function $[x] = x - \{x\}$ in Eq. (54) allows writing $R = R_M + R_E$ in terms of a main and error term

$$\begin{aligned} R_M &= R_0 + \sum_{p_j < n} \mu(n) 2^{\nu(n)} \frac{L(p_j) - M(j+1) - 1}{2n}, \\ R_E &= - \sum_{p_j < n} \mu(n) 2^{\nu(n)} \left\{ \frac{L(p_j) - M(j+1) - 1}{2n} \right\}. \end{aligned} \quad (58)$$

Theorem 5.4. *The main term R_M in Eq. (54) satisfies*

$$\begin{aligned} R_M &= \frac{1}{2} L(p_j) \prod_{5 \leq p \leq (6L(p_j)+1)/4} \left(1 - \frac{2}{p} \right) \\ &+ \frac{1}{2} M(j+1) \left[1 - \prod_{p_j < p \leq (6L(p_j)+1)/4} \left(1 - \frac{2}{p} \right) \right]. \end{aligned} \quad (59)$$

Proof. Expanding the product

$$L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p} \right) \quad (60)$$

and combining corresponding sums in Eq. (58)

$$- \sum_{5 \leq p \leq p_j} \frac{1}{p} - \sum_{p_j < p \leq (6L(p_j)+1)/4} \frac{1}{p} = - \sum_{5 \leq p \leq (6L(p_j)+1)/4} \frac{1}{p}, \dots \quad (61)$$

just shifts the upper limit of the primes in the product $\prod_p(1 - 2/p)$ from p_j to $(6L(p_j) + 1)/4$, so that we obtain Eq. (59). This involves considerable cancellations collapsing R_0 to the correct magnitude of R_M . \diamond

Theorem 5.5. *The main term R_M obeys the asymptotic law*

$$R_M \sim \frac{2c_2 e^{-2\gamma} 3L(p_j)}{\log^2((3L(p_j) + 0.5)/2)}, \quad p_j \rightarrow \infty. \quad (62)$$

Proof. This follows as Theor. 5.8 in Ref. [5]. \diamond

6 Summary and Discussion

The cousin prime sieve is specifically designed for prime twins at distance 4, often called cousin primes.

Accurate counting of non-rank sets require the infinite, but sparse set of odd 'primorials' $\{3L(p_j) = \prod_{2 < p \leq p_j} p\}$ much like in other sieves when applied to primes. The twin-4 primes are not directly sieved, rather twin-4 ranks $2m+1$ are with $3(2m+1) \pm 2$ both prime. All other odd numbers (≥ 9) are non-ranks. Primes serve to organize and classify non-ranks in arithmetic progressions with equal distances (periods) that are primes or products of them leading to the (odd) primorials.

The coefficient $4c_2 e^{-2\gamma} \approx 0.8324267$ of the asymptotic form of the main term is the same as for ordinary twins, despite the differences in the arithmetic of the pair sieves. Just as for ordinary twin primes, the resolution of the parity problem allows replacing the need for a lower bound on R , or π_2 at primorials, by an upper bound for the error term R_E .

References

- [1] H. Halberstam and H. E. Richert, Sieve Methods, Acad. Press, New York, 1974; Dover, New York (2011).
- [2] M. Ram Murty, Problems in Analytic Number Theory, Springer, New York (2001).
- [3] H. Riesel, Prime Numbers and Computer Methods for Factorization, 2nd ed., Birkhäuser, Boston (1994).

- [4] J. Friedlander and H. Iwaniec, Opera de Cribro, Amer. Math. Soc. Colloq. Publ. **59** (2010), Prov. RI.
- [5] A. Dinculescu and H. J. Weber, *Twin Prime Sieve*, www.arXiv.org/1203.5240
- [6] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Clarendon Press, Oxford, 5th ed. (1988).
- [7] H. J. Weber, *Regularities of Prime Number Twins, Triplets and Multiplets*, Global J. Pure a. Applied Math. **8** (2012), www.adsabs.harvard.edu/abs/2011 arXiv1103.0447W